The Fano of Lines and the Kuznetsov component of cubic 4-folds
(Joint with Ed Segal)

— $Y = V(f) \subseteq \mathbb{P}^5$ $\quad$ $f \in H^0(\mathbb{P}^5, \mathcal{O}(3))$ $\quad$ (/$\mathbb{C}$ smooth)
cubic 4-fold

$f \in H^0(\mathrm{Gr}(2,6), \mathrm{Sym}^3 \mathcal{U}^*)$ $\quad$ $\mathcal{U}$ taut rank 2

⇝ rationality: (Derived cat. perspective)

$D^b(Y)$ cat. bounded complexes of v. bundles on Y

For $\mathbb{P}^5$ $\quad$ $D^b(\mathbb{P}^5) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(5) \rangle$

generate $D^b(\mathbb{P}^5)$
$\oplus$, extension, shifts.
Full exceptional collection.

$X \to \mathbb{C}$ $\begin{cases} 3 \text{ odp} \\ \cdot\ K3 \end{cases}$

$D^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$

Kuznetsov component.
wants $D^b(K3)$

" non-comm. K3 "
- Serre duality [2]
- Has "same" $HH_\bullet$ as $D^b(K3)$

Hodge theoretic version (Hassett) ⇝ <u>Kuznetsov's conj</u>: Y is rat $\Longleftrightarrow$ $\mathcal{A}_Y \cong D^b(K3)$

"$\mathcal{A}_Y$ is geometric"

Y cub

"non-comm" K3 $\mathcal{A}_Y$

Suppose $\mathcal{A}_Y \cong D^b(S)$ K3.

$\underline{S^{[2]}}$

<u>Hyperkähler 4-fold.</u> (BD '85)

F_Y Fano of lines of K3-2 type

$F_Y \subseteq Gr(2,6)$

$\| \; \mathcal{Z}(f)$

"symmetric square"

$\underset{\sim}{\mathcal{A}_Y^{[2]}}$ (<u>geom</u>: $D^b(S)^{[2]} \underset{BKR}{=} D^b(S^{[2]})$)

<u>non-comm HK 4-fold</u>

<u>Conj (Galkin)</u> $\mathcal{A}_Y^{[2]} \cong D^b(F_Y)$

- true at level of <u>K_0(dg cat)</u> (Galkin-Shinder)
- Belmans-Fu-Raedschelders

$D^b(Y^{[2]}) = \langle \; , \; \underbrace{\qquad}_{6 \text{ copies}} \rangle$

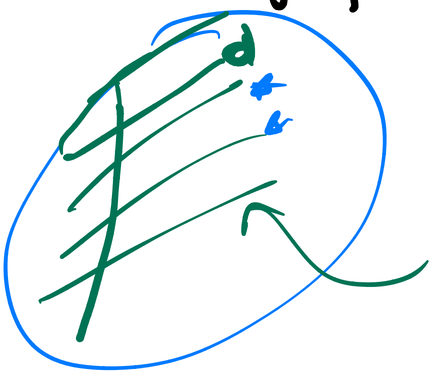
Might hope: $\mathcal{A}_Y \simeq D^b(S)$ earlier $\qquad \mathcal{A}_Y^{[2]}$ / $D^b(F_Y)$ $\qquad$ of $D^b(Y)$

(hard!) $\quad$ If $S^{[2]} \sim F_Y$ then ✓

D-eq. for K3[2]-type.

Thm Bottini – Huybrechts (Jan 25)

Prove conj for cubics s.t. F_Y admits a rational Lag fibration.



$H^0(\mathbb{P}^5, \mathcal{O}(3)) \,/\, GL(6, \mathbb{C})$

cubics

Y s.t general in $\mathcal{C}_d$

s.t. $\frac{d}{2}$ perfect square.

Thm (K. – Segal) Conj is true for all smooth cubic 4-folds

(From a non-comm. perspective)

II: Matrix factorisations

Setup: (X, W) X (smooth) scheme / alg stack.

$W \in H^0(X, \mathcal{O}_X)$ "superpotential"

"A matrix factorization": a bounded "complex" of v. bundles $\mathcal{E}^\bullet$ but $d^2 = \underline{W} \cdot \mathrm{Id}$

eg: $(X, 0)$ MF $\leftrightsquigarrow$ complex.

- $MF(X, W)$ cat. whose objects are MF. $\left(MF(X,0) \cong D^b(X) \right)$

Knörrer periodicity X smooth scheme, E vector bundle.

$s \in H^0(X, E)$ $W_s : \mathrm{Tot}(E) \to \mathbb{C}$

(assume $V(s)$ has exp. dim)

$$MF(\mathrm{Tot}(E), W_s) \cong D^b(V(s))$$

$$A_Y^{[2]} \cong D^b(F_\epsilon)$$

$\boxed{\text{app:}}$ $D^b(F_Y) \cong MF(\mathrm{Sym}^3 U, W)$

$\mathrm{Sym}^3 U \downarrow Gr(2,6)$

W induced f

<u>Orlov (Segal)</u>

let $V = \mathbb{C}^n \times \mathbb{C} \circlearrowleft \mathbb{G}_m$

$x_1, \dots, x_n$; y

$1, \dots, 1$; $\cancel{-n}$ $-d$

f hom. $\cancel{\deg n}$ d in $x_1, \dots, x_n$.

$W = fy$

$$X_- = \frac{(\mathbb{C}^n \setminus 0) \times \mathbb{C}}{\mathbb{G}_m} = \mathrm{Tot}(\mathcal{O}(\cancel{n})(d)) \longleftrightarrow X_+ = \frac{\mathbb{C}^n \times (\mathbb{C} \setminus 0)}{\mathbb{G}_m} = \mathbb{C}^n / \mathbb{Z}_d$$

$\downarrow$

$(\mathbb{C}^n \setminus 0)/\mathbb{G}_m \cong \mathbb{P}^{n-1}$

$$X_- \hookrightarrow V/\mathbb{G}_m \hookleftarrow X_+$$

If $d < n$

$D^b(V/\mathbb{G}_m)$

$n=6$
$\deg(f)=3$

$$D^b(X_-) \xleftarrow{\ \simeq\ } \underbrace{\langle \mathcal{O}, \dots, \mathcal{O}(n-1) \rangle}_{\text{"window category"}} \xrightarrow{\ \simeq\ } D^b(X_+)$$

$$D^b(V(f)) = MF(X_-, W) \simeq MF(X_+, W)$$

$$D^b(V(f)) = MF(X_-, W) \simeq \langle MF(X_+, W), \mathcal{O}, \dots, \mathcal{O}(n-d-1) \rangle$$

$$D^b(Y) = \langle MF(\mathbb{C}^6/\mathbb{Z}_3, f), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$$

$$\underset{A_Y}{\parallel}$$

$$A_Y^{[2]} = MF\left(\frac{\mathbb{C}^6 \times \mathbb{C}^6}{(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2} , f_1 + f_2 \right)$$

$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2 = \Gamma$

Conj:

$$\mathrel{\wr\!\wr} ?$$

$$D^b(F_Y) = MF(\mathrm{Sym}^3(U), W)$$

induced from f.

The GIT problem:

$$V = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6) \times \mathrm{Sym}^3(\mathbb{C}^2) \circlearrowleft Gl_2$$

$$X_- = \left(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6)^{\text{full rank}} \times \mathrm{Sym}^3(\mathbb{C}^2)\right) /\!\!/ Gl_2 = \mathrm{Tot}(\mathrm{Sym}^3 \mathcal{U}^*) \xleftarrow[\text{12 dim}]{\text{"flop"}} X_+ \quad \text{orbifold}$$

$$= \mathrm{Sym}^3 \mathcal{U}.$$

$$\downarrow$$

$$\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6)^{\text{full rank}} / Gl_2 \cong Gr(2,6)$$

$$X_+ \supseteq X^0$$

$$X^0 = \frac{\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6) \times \left(\begin{matrix}\text{cubics w}\\ \text{distinct}\\ \text{roots}\end{matrix}\right)}{Gl_2}$$

$$t_1^3 + t_2^3$$

$$\frac{\mathbb{C}^6 \times \mathbb{C}^6}{\Gamma} = \frac{\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^6) \times \left(\begin{matrix}\text{cubics w}\\ \text{distinct}\\ \text{roots}\end{matrix}\right)}{Gl_2}$$

Thm. (K-Segal) $D^b(X_-) \cong D^b(X_+)$

$$D^b(F_Y) \cong MF(X_-, W) \overset{\text{Thm}}{\cong} MF(X_+, W) \xrightarrow{\cong} MF(X^\circ, f_1 + f_2) \cong \mathcal{A}_Y^{[2]}$$

ORlov: $MF(X, W) \rightsquigarrow$ "singularity cat" $W^{-1}(0)$

Thm2 If Y is smooth, then $\cong$

f nonsing then $crt(W)$ in X^+ is contained in X°